Problem Set 4

This fourth problem set explores set cardinality and graph theory. We figure that this will serve as a cool, whirlwind tour of the infinite (through set theory) and the finite (through graphs and their properties) and will give you a better sense for how discrete mathematical structures connect across these domains.

Before starting this problem set, please read the online Guide to Cantor's Theorem, which contains several important terms and definitions you'll need on this problem set, along with a formal proof of Cantor's theorem.

Good luck, and have fun!

Checkpoint due Monday, May 1 at the start of lecture.

Remaining problems due Friday, May 5 at the start of lecture.

These problems should be completed and submitted before Monday's lecture.

Checkpoint Problem: A Really Simple Bijection? (2 Points if Submitted)

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = n$.

i. Prove that f is not a bijection.

Below is a purported proof that f is a bijection:

> ***Theorem:*** Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as $f(n) = n$. Then f is a bijection.
>
> ***Proof:*** In lecture, we proved that $|\mathbb{N}| = |\mathbb{Z}|$. Since the sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, we know that every function between them must be a bijection. In particular, this means that f must be a bijection, as required. ■

This proof has to be incorrect, since, as you proved in part (i), f isn't a bijection.

ii. What's wrong with this proof? Justify your answer.

The remaining problems on this problem set are due on Friday, May 5th at the start of lecture.

Problem One: Set Cardinalities

Let a and b be arbitrary objects such that $a \neq b$. Using the formal definition of equal cardinalities, we'd like you to prove that $|\mathbb{N} \times \{a, b\}| = |\mathbb{N}|$, and we've broken this task down into three steps.

i. Draw a picture showing a way to pair off the elements of $\mathbb{N} \times \{a, b\}$ with the elements of $\mathbb{N}$.

ii. Based on the picture you came up with in part (i), define a bijection $f : \mathbb{N} \times \{a, b\} \to \mathbb{N}$.

iii. Prove that the function you came up with in part (ii) is a bijection.

Problem Two: Understanding Diagonalization

Proofs by diagonalization are tricky and rely on nuanced arguments. In this problem, we'll ask you to review the diagonalization proof we covered in the Guide to Cantor's Theorem to help you better understand how it works. (Please read the Guide to Cantor's Theorem before attempting this problem.)

i. Consider the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as $f(n) = \emptyset$. Trace through our proof of Cantor's theorem with this choice of f in mind. In the middle of the argument, the proof defines some set D in terms of f. Given that $f(n) = \emptyset$, what is that set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

ii. Let f be the function from part (i). Find a set $S \subseteq \mathbb{N}$ such that $S \neq D$, but $f(n) \neq S$ for any $n \in \mathbb{N}$. Justify your answer. This shows that while the diagonalization proof will always find *some* set D that isn't covered by f, it won't find *every* set with this property.

iii. Repeat part (i) of this problem using the function $f : \mathbb{N} \to \wp(\mathbb{N})$ defined as

$$f(n) = \{ \ m \in \mathbb{N} \mid m \geq n \ \}$$

Now what do you get for the set D? Is it clear why $f(n) \neq D$ for any $n \in \mathbb{N}$?

iv. Repeat part (ii) of this problem using the function f from part (iii).

Problem Three: Simplifying Cantor's Theorem?

In our proof of Cantor's theorem, we proved that $|S| \neq |\wp(S)|$ by using a diagonal argument. Below is a purported proof that $|S| \neq |\wp(S)|$ that doesn't use a diagonal argument:

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be any set and consider the function $f : S \to \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.

Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal if and only if their elements are the same, this means that $x_1 = x_2$, as required.

However, f is not surjective. Notice that $\emptyset \in \wp(S)$, since $\emptyset \subseteq S$ for any set S, but that there is no x such that $f(x) = \emptyset$; this is because $\emptyset$ contains no elements and $f(x)$ always contains one element. Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$. ■

Unfortunately, this proof is incorrect. What's wrong with this proof? Justify your answer.

Problem Four: Paradoxical Sets

What happens if we take *absolutely everything* and throw it into a set? If we do, we would get a set called the ***universal set***, which we denote $\mathcal{U}$:

$$\mathcal{U} = \{ x \mid x \text{ exists} \}$$

Absolutely everything would belong to this set: $1 \in \mathcal{U}$, $\mathbb{N} \in \mathcal{U}$, philosophy $\in \mathcal{U}$, CS103 $\in \mathcal{U}$, etc. In fact, we'd even have $\mathcal{U} \in \mathcal{U}$, which is strange but not immediately a problem.

Unfortunately, the set $\mathcal{U}$ doesn't actually exist, as its existence would break mathematics.

i. Prove that if A and B are arbitrary sets where $A \subseteq B$, then $|A| \le |B|$. *(Hint: Look at the Guide to Cantor's theorem. Formally speaking, if you want to prove that* $|A| \le |B|$*, what do you need to prove? Your answer should involve defining some sort of function between A and B and proving that function has some specific property or properties.)*

ii. Using your result from (i), prove that if $\mathcal{U}$ exists, then $|\wp(\mathcal{U})| \le |\mathcal{U}|$.

iii. Using your result from (ii) and Cantor's Theorem, prove that $\mathcal{U}$ does not exist. Feel free to the following fact in the course of writing up your proof: for any sets A and B, if $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.

The result you've proven shows that there is a collection of objects (namely, the collection of everything that exists) that cannot be put into a set. When this was discovered at the start of the twentieth century, it caused quite a lot of chaos in the math world and led to a reexamination of logical reasoning itself and a more formal definition of what objects can and cannot be gathered into a set. If you're curious to learn more about what came out of that, take Math 161 (Set Theory) or Phil 159 (Non-Classical Logic).

Problem Five: Independent Sets and Vertex Covers

An ***independent set*** in a graph $G = (V, E)$ is a set $I \subseteq V$ with the following property:

$$\forall u \in I.\ \forall v \in I.\ \{u, v\} \notin E.$$

i. Translate the definition of an independent set into plain English. No justification is necessary.

ii. You want to conduct a poll for an election. You're interested in polling people who are all complete strangers to one another so that you can get a good representative sample of the population, and you'd like to find a large group of mutual strangers so that your poll has good statistical power. Explain how you might model this problem in terms of building some sort of graph, then finding a large independent set in it. Briefly justify your answer; no formal proof is necessary.

The size of an independent set is the number of nodes in it. Formally speaking, if I is an independent set, then the size of I is given by $|I|$.

iii. Draw a graph with five nodes that contains an independent set of size five. Briefly justify your answer; no formal proof is necessary.

iv. Draw a graph with five nodes that contains no independent sets of size two or greater. Briefly justify your answer; no formal proof is necessary.

A ***vertex cover*** in a graph $G = (V, E)$ is a set $C \subseteq V$ with the following property:

$$\forall u \in V.\ \forall v \in V.\ (\{u, v\} \in E \rightarrow u \in C \lor v \in C).$$

The remainder of this problem explores vertex covers and their relation to independent sets.

v. Translate the definition of a vertex cover into plain English. No justification is necessary.

vi. Suppose that you have a map of a city's roads and streets (assume that the roads are set up so that it's possible to walk between any two points in the city). You want to set up information kiosks so that no matter where someone is, they need to walk at most one block to find an information kiosk. You also want to use as few information kiosks as possible to accomplish this. Explain how you might model this problem by building some sort of graph and looking for a small vertex cover in that graph. Briefly justify your answer; no formal proof is necessary.

The size of a vertex cover is the number of nodes in it. Formally speaking, if C is a vertex cover, then the size of C is given by $|C|$.

vii. Draw a graph with five nodes that has a vertex cover of size zero. Briefly justify your answer; no formal proof is necessary.

viii. Draw a graph with five nodes that has a vertex cover of size four and no vertex covers of size three or less. Briefly justify your answer; no formal proof is necessary.

There's a close connection between independent sets and vertex covers.

ix. Prove that if $G = (V, E)$ is a graph and I is a an independent set in G, then $V - I$ is a vertex cover of G.

x. Prove that if $G = (V, E)$ is an arbitrary graph and C is a vertex cover of G, then $V - C$ is an independent set in G.

The connection between independent sets and vertex covers that you proved in the last two parts of this problem show that the problem of finding a maximum independent set (an independent set that's as large as possible) in a graph is essentially the same as finding a minimum vertex cover (a vertex cover that's as small as possible) in that graph. Both of these problems are known to be **NP**-hard, so we suspect they're computationally infeasible to solve for large graphs. We'll talk more about **NP**-hardness in week ten.

Problem Six: Bipartite Graphs

The ***bipartite graphs*** are a special class of graphs with applications throughout computer science. An undirected graph $G = (V, E)$ is called ***bipartite*** if there exists two sets V_1 and V_2 such that

- every node $v \in V$ belongs to exactly one of V_1 and V_2, and
- every edge $e \in E$ has one endpoint in V_1 and the other in V_2.

To help you get a better intuition for bipartite graphs, let's consider an example. Suppose that you have a group of people and a list of restaurants. You can illustrate which people like which restaurants by constructing a bipartite graph where V_1 is the set of people, V_2 is the set of restaurants, and there's an edge from a person p to a restaurant r if person p likes restaurant r.

Bipartite graphs have many interesting properties. One of the most fundamental is this one:

An undirected graph is bipartite if and only if it contains no cycles of odd length.

Intuitively, a bipartite graph contains no odd-length cycles because cycles alternate between the two groups V_1 and V_2, so any cycle has to have even length.

The trickier step is proving that if G contains no cycles of odd length, then G has to be bipartite. For now, assume that G has just one connected component; if G has multiple connected components, we can treat each one as a separate graph for the purposes of determining whether G is bipartite. (You don't need to prove this, but I'd recommend taking a minute to check why this is the case.)

Suppose G is a connected, undirected graph with no cycles of odd length. Choose any node $v \in V$. Let V_1 be the set of all nodes that are connected to v by a path of odd length and V_2 be the set of all nodes connected to v by a path of even length. (Note that these paths do not have to be simple paths). Formally:

$$V_1 = \{ x \in V \mid \text{there is an odd-length path from } v \text{ to } x \}$$

$$V_2 = \{ x \in V \mid \text{there is an even-length path from } v \text{ to } x \}$$

i. Prove that V_1 and V_2 have no nodes in common.

ii. Using your result from part (i), prove that if G is connected and has no cycles of odd length, then G is bipartite. *(Hint: look back at the definition of a bipartite graph. What exactly do you need to prove to show that a graph is bipartite?)*

Problem Seven: Jelly Bean Jars

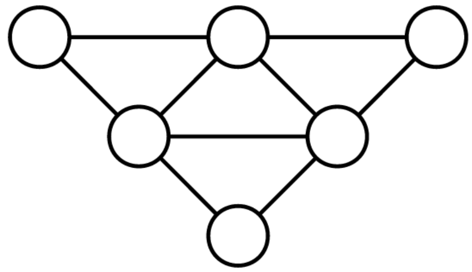
Suppose that there are six flavors of jelly beans and that you have eleven jelly beans of each flavor. You distribute those jelly beans across five jars. Prove that no matter how you distribute them, there will always be a jar with at least three jelly beans of one flavor and at least three jelly beans of a different flavor.

(Giving credit where credit is due: this excellent pigeonhole principle problem comes from a problem set given at MIT. I just thought it was such a good problem that I couldn't pass up on it. ☺)

Problem Eight: Chromatic and Independence Numbers

Recall from lecture that the ***chromatic number*** of a graph G, denoted $\chi(G)$, is the minimum value of k for which the graph has a k-coloring. A related concept is the ***independence number*** of a graph G, denoted $\alpha(G)$, which is the size of the largest independent set in G.

Consider the following graph G:



i. What is $\chi(G)$? What is $\alpha(G)$? No justification is required.

ii. Let n be an arbitrary positive natural number. Prove that if G is an undirected graph with exactly n^2+1 nodes, then $\chi(G) \geq n+1$ or $\alpha(G) \geq n+1$ (or both). As a hint, look at Handout 14's advice about how to prove a statement of the form $P \lor Q$.

Extra Credit Problem: Hugs All Around! (1 Point Extra Credit)

There's a party with 137 attendees. Each person is either ***honest***, meaning that they *always* tell the truth, or ***mischievous***, meaning that they *never* tell the truth.

After everything winds down, everyone is asked how many honest people they hugged at the party. Surprisingly, each of the numbers 0, 1, 2, 3..., and 136 was given as an answer exactly once.

How many honest people were at the party? Prove that your answer is correct and that no other answer could be correct.